COSC201: Tutorial Weeks 2 and 4
Proofs: Induction

1 Exercises on Induction

1. Prove that $2n + 1 = O(2^n)$. Remember to use induction!

   Model Answer:

   Proof. Show that $2n + 1 \leq c.2^n$ for $n \geq n_0$. Choose $c = 1, n_0 = 3$.

   Base case: $LHS = 2.3 + 1 = 7 \leq 2^3 = 8 = RHS$.

   Inductive step: Assume that $2k + 1 \leq 2^k$, show that $2(k + 1) + 1 \leq 2^{k+1}$ for $k \geq 3$:

$$\begin{aligned} LHS &= 2(k+1) \\ &= 2k + 2 + 1 \\ &= 2k + 1 + 2 \\ &\leq 2^k + 2 \\ &\leq 2^k + 2^k \\ &\leq 2^{k+1} \\ &\leq RHS \end{aligned}$$

   □

2. Prove that $n^2 = O(n!)$. Remember to use induction!

3. Prove that $1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6$ for all $n \geq 1$.

   Model Answer:

   Proof. Base case: $LHS = 1; RHS = 1.2.3/6 = 1 = LHS$

Inductive step: Assume that $1^2 + 2^2 + \ldots + k^2 = k(k+1)(2k+1)/6$, show that $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6$.

$$\begin{aligned} LHS &= k(k+1)(2k+1)/6 + (k+1)^2 \\ &= (k+1)(k(2k+1) + 6(k+1))/6 \\ &= (k+1)(2k^2 + k + 6k + 6)/6 \\ &= (k+1)(2k^2 + 7k + 6)/6 \\ &= (k+1)(k+2)(2k+3)/6 \\ &= RHS \end{aligned}$$

□

4. Prove that $4^n - 1$ is divisible by 3 for all $n \geq 1$.

Model Answer:

Proof. Base case: $LHS = 4^1 - 1 = 3 = 3.1 = 3 \times 1 = RHS$

Inductive step: Assume that $4^k - 1 = 3m$ for some m, show that $4^{k+1} - 1 = 3n$ for some n.

$$\begin{aligned} LHS &= 4^{k+1} - 1 \\ &= 4.4^k - 1 \\ &= 4.4^k - 4 + 3 \text{ because } -4 + 3 = 1 \\ &= 4(4^k - 1) + 3 \\ &= 4.3m + 3 \\ &= 3(4m + 1) \\ &= 3n \text{ for } n = 4m + 1 \end{aligned}$$

And therefore, by induction, $4^n - 1$ is divisible by 3 for all $n \geq 1$. □