COSC242: Tutorial Week 4
Proofs: Substitution and Iteration plus

1 Some worked answers

1. Suppose $\begin{array}{lcl} g(1) & = & 0 \\ g(n) & = & g(n-1) + (n-1) \end{array}$

   Solve by iteration and prove your solution correct by induction.

   Model Answer: Iteration and substitution

$$\begin{aligned} g(n) &= g(n-1) + (n-1) \\ &= g(n-2) + (n-1-1) + n - 1 \\ &= g(n-2) + 2n - 2 - 1 \\ &= g(n-3) + (n-2-1) + 2n - 2 - 1 \\ &= g(n-3) + 3n - 3 - 2 - 1 \\ &= g(n-4) + 4n - 4 - 3 - 2 - 1 \\ &= g(n-k) + kn - k - (k-1) - ... - 1 \\ &= g(n-k) + kn - \frac{k(k+1)}{2} \end{aligned}$$

Set $k = n - 1$

$$\begin{aligned} &= g(1) + (n-1)n - \frac{(n-1)n}{2} \\ &= \frac{(n-1)n}{2} \end{aligned}$$

Induction proof Let $f(n) = \frac{(n-1)n}{2}$, and show that $g(n) = f(n)$.

Proof. Base case: $g(1) = 0 = f(1)$

Inductive step: Assume that $g(k) = f(k)$ and show that $g(k+1) = f(k+1)$.

$$\begin{aligned}
g(k+1) &= g(k) + (k+1-1) \\
&= f(k) + k \\
&= \frac{(k-1)k}{2} + k \\
&= \frac{(k-1)k + 2k}{2} \\
&= \frac{k^2 - k + 2k}{2} \\
&= \frac{k^2 + k}{2} \\
&= \frac{k(k+1)}{2} \\
&= f(k+1)
\end{aligned}$$

$\square$

2. Suppose $\begin{array}{lcl} f(1) & = & 1 \\ f(n) & = & n \cdot f(n-1) \end{array}$

   Solve by iteration and prove your solution correct by induction.

Model Answer: Iteration and substitution

$$\begin{aligned}
f(n) &= n \cdot f(n-1) \\
&= n \cdot (n-1) \cdot f(n-2) \\
&= n \cdot (n-1) \cdot (n-2) \cdot f(n-3) \\
&= n \cdot (n-1) \cdot (n-2) \ldots \cdot 2 \cdot 1 \\
&= n!
\end{aligned}$$

Induction proof Let $g(n) = n!$ and show that $f(n) = g(n)$.

Proof. Base case: $f(1) = 1 = g(1)$
Inductive step: Assume that $f(k) = g(k)$ and show that $f(k+1) = g(k+1)$.

$$\begin{aligned}
f(k+1) &= (k+1) \cdot f(k) \\
&= (k+1) \cdot g(k) \\
&= (k+1) \cdot k! \\
&= (k+1)! \\
&= g(k+1)
\end{aligned}$$

$\square$

3. We have proven in lectures that Mergesort's time complexity is $n \log n + n$. Prove that $n \log n + n = O(n \log n)$.

Model Answer:

Proof. Show that $n \log n + n \leq cn \log n$ for $n \geq n_0$. Choose $c = 2, n_0 = 2$:

$$\begin{aligned} \text{LHS} &= n \log n + n \\ &\leq n \log n + n \log n \text{ if } n \geq 2 \\ &\leq 2n \log n \\ &\leq \text{RHS} \end{aligned}$$

□

4. Prove that $1^3 + 2^3 + \ldots + n^3 = [n(n+1)/2]^2$ for all $n \geq 0$.

Model Answer:

Proof. Base case: $LHS = 0; RHS = 0^2 = 0 = LHS$

Inductive step: Assume that $1^3 + 2^3 + \ldots + k^3 = [k(k+1)/2]^2$, show that $1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = [(k+1)(k+2)/2]^2$.

$$\begin{aligned} LHS &= [k(k+1)/2]^2 + (k+1)^3 \text{ using the assumption} \\ &= (k+1)^2[k^2/2^2 + (k+1)] \text{ factor out } (k+1)^2 \\ &= (k+1)^2[(k^2 + 4k + 4)/4] \\ &= (k+1)^2[(k+2)^2/4] \\ &= [(k+1)(k+2)/2]^2 \\ &= RHS \end{aligned}$$

□

5. Prove that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for all $n \geq 0$.

Model Answer:

Proof. Base case: $LHS = 6^2 + 7 = 43$ which is divisible by 43.

Inductive step: Assume that $6^{k+2} + 7^{2k+1} = 43m$ for some m, show that $6^{k+1+2} + 7^{2k+1+2} = 43n$ for some n.

$$
\begin{aligned}
LHS &= 6^{k+1+2} + 7^{2k+1+2} \\
&= 6.6^{k+2} + 7^2.7^{2k+1} \text{ take a 6 out of the first term, and } 7^2 \text{ out of the second} \\
&= 6.6^{k+2} + 49.7^{2k+1} \\
&= 6.6^{k+2} + 43.7^{2k+1} + 6.7^{2k+1} \text{ split the 49 into two parts.} \\
&= 6.(6^{k+2} + 7^{2k+1}) + 43.7^{2k+1} \text{ gather the multiples of 6.} \\
&= 6.m.43 + 43.7^{2k+1} \\
&= 43(6m + 7^{2k+1}) \\
&= 43n \text{ with } n = 6m + 7^{2k+1} \\
&\text{which is divisible by 43.}
\end{aligned}
$$

$\square$