Cosc 201
Algorithms and Data Structures
Lecture 17 (29/4/2026)
Graph Paths

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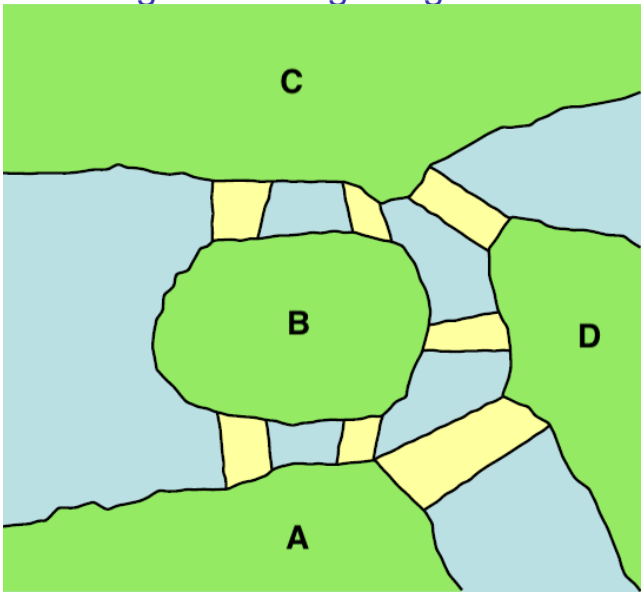
Seven Bridges of Königsberg



Paths in graphs

- One of the standard metaphors in graph theory is to think of the edges as physical connections between locations represented by their endpoints.
- Indeed, the Seven Bridges of Königsberg problem resolved in 1736 by Euler "laid the foundations of graph theory".
- A path in a simple graph is a sequence of *distinct* vertices $v_0, v_1, \ldots, v_k$ such that every consecutive pair of them (i.e., v_0 and v_1, v_1 and v_2, $\ldots$, v_{k-1} and v_k) are connected by an edge.
- The length of that path is k - the number of *edges* it uses (in particular, paths of length 0 exist and are just vertices!)
- A walk is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that every consecutive pair of them are connected by an edge. The vertices need not be distinct.

Connectedness again

- A graph, G, is connected if, given any two vertices v and w of G there is a *walk* from v to w.
- If we make a union-find instance on the vertices of G and then do a union operation for the endpoints of every edge, then that's equivalent to saying that all the vertices belong to the same group (i.e., there's a single representative) at the end.
- Is it? That should probably be proven.
- More interesting, is it also true that if there's a walk from v to w there's necessarily a path?
- Yes! If there is a walk, then there's a shortest walk, and the shortest walk must be a path.
- Because, if a walk contains some vertex, z, two or more times, then we can cut out the parts between the first and last occurrences of z (replacing them by a single occurrence) and get a shorter walk. So a shortest walk contains no repeated vertices i.e., it's a path.

Finding shortest paths

That highlights an interesting problem:

Single-pair shortest path problem

Given a graph G and vertices v and w find a shortest path from v to w (or show that no path exists).

More generally:

Single-source shortest path problem

Given a graph G and a vertex v find shortest paths from v to every vertex that it can reach.

Hint: We already know how to do this!

Breadth-first traversal revisited

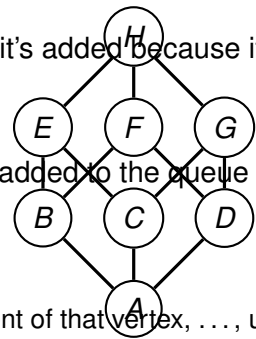
- The distance between two vertices is the length of the shortest path between them.
- In a BFT in G from v we first add v to a queue.
- That's the only vertex at distance 0 from v.
- Then we add all its neighbours - definitely the vertices at distance one.
- Then we add *their* neighbours *unless already seen*, these are clearly at distance two!
- And that "obviously" persists.
- So we could certainly compute an array of distances – but we want *paths*.

Dijkstra's shortest-path algorithm (unweighted case)

- Start a BFT from the source vertex v.
- At the moment when a vertex x is added to the queue it's added because it's a neighbour of some vertex x.
- Record this information $y.parent \leftarrow x$.
- Stop as usual when the queue is empty (or when w is added to the queue if you're only interested in a path from v to w).
- To find the shortest path from v to (any vertex) w:
  - If $w.parent$ is not defined, there isn't one.
  - Otherwise, read backwards w, $w.parent$, then the parent of that vertex, ..., until you reach v.
- We've added a constant amount of work in the vertex-processing part of a BFT so the complexity is still $O(|V| + |E|)$.

Consider the graph as shown. Which of the following is a breadth-first traversal of the graph starting at node A?

1. ABCDEFGH
2. ABEHFGDC
3. ABEFCGDH



Dijkstra's shortest-path algorithm (weighted case)

- In most shortest-path problems the edges are *weighted* and/or directed and the length (or weight) of a path is defined to be the sum of the weights of its edges.
- If the weights are always positive, can we cope?
- What about the absolutely shortest (non-zero) path from v.
- It must consist of a single edge (because the weights are positive).
- In fact it must be the edge of least weight that starts from v.
- In a basic BFT its other endpoint will be one of the ones added in the first round of additions to the queue - but we probably want to look at it next.
- What kind of queue allows removal based on some other criterion?
- We can replace the standard queue with a min-priority queue!

Dijkstra's shortest path algorithm

```
1:  procedure DIJKSTRA(G, v)
2:      dist[v] ← 0
3:      for each edge e = (v, u) from v do
4:          q.add(e, e.weight)
5:      end for
6:      while q is not empty do
7:          e = (x, y) ← q.remove()
8:          if y.dist is not known then
9:              dist[y] ← dist[x] + e.weight
10:             y.parent ← x
11:             for each edge e = (y, z) from y do
12:                 if dist[z] is not known then
13:                     q.add(e, dist[y] + e.weight)
14:                 end if
15:             end for
16:         end if
17:     end while
18: end procedure
```

Weighted Dijkstra correctness

- We know the *weight* of the least-weight paths to some vertices (initially just the base).
- We have a min-priority queue whose keys are edges representing the last edge of a potential path. The source of that edge will always have a known weight.
- The priority associated to the key is the weight of its source plus the weight of the edge itself.
- When we poll the queue, the edge returned will have a target which is either:
  - Already of known weight, in which case we just continue, or
  - Of unknown weight, in which case we can compute and set its weight, assign its parent, and add all the outedges from it to the queue.
- We will look in more detail next time.