Cosc 201
Algorithms and Data Structures
Lecture 13 (7/4/2025)
Two proofs

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Proving connection and union-find

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Actually we'll prove something stronger.

Connection and union-find

In a graph, G there is a walk from v to w if and only if the union-find instance formed by starting with the vertices of G and taking the union between the endpoints of each edge v and w belong to the same group, i.e., $\mathit{find}(v) = \mathit{find}(w)$.

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- To do this, we'll show that at any point during the process of doing the union operations and for any vertex v there is always a walk from v to any other vertex u such that $find(v) = find(u)$, i.e., to any other vertex in the same group as v at that point.

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- This will be by induction over the sequence of union operations performed.
- This is certainly true before the first union operation since $\mathit{find}(v) = \mathit{find}(u)$ means $v = u$ before the first *union* operation.

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- Yes! Because we could already walk to x (and any vertex in its original group), and now we can follow the edge from there to y and then from y to any vertex in *its* original group.
- So, by induction, the property holds after each and every *union* operation (including the last one, which is when we need it).

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- To find the set of vertices, c we can reach from v by walks we just modify the BFT a bit (see next slide).
- In common graphics contexts (where we are working in a graph which is a grid) variations on this idea include the flood fill algorithm.

BFT to find the vertices reachable from *v*

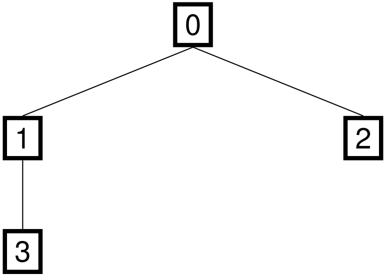
q: an initally empty queue.
c: an initially empty set of vertices.

```
q.addAll(v.neighbours), c.add(v)
while q is not empty do
    w ← q.remove(), c.add(w)
    for n in w.neighours do
        if n is not in c then
            q.add(n), c.add(n)
        end if
    end for
end while
```

Quiz question

Which of the following orderings of the vertices is a valid breadth-first traversal of the given graph starting at node 0?

1. 0, 1, 2, 3
2. 0, 1, 3, 2
3. 0, 3, 2, 1
4. 0, 3, 1, 2



The weighted shortest-path algorithm works?

The data structures associated with the weighted single-source (*base*) shortest-path algorithm are:

weights A map from vertices, *w*, to integers that represents the known weight of the least-weight path from *base* to *w*. Vertices are supposed to be added to this map in increasing order of weight.

parent A map from vertices, *w*, to vertices that represents the parent of *w* along the least-weight path from *base* to *w*.

pq A min-priority queue of edges. The source of each edge in *pq* is a vertex of known weight, and the priority of the edge is the sum of that weight and the weight of the edge.

Our job is to analyse the process of the algorithm and see that at all times these conditions are satisfied.

Initialisation

The intialisations are:

weights	The weight of *base* is set to 0.
parent	The parent of *base* is set to some “end of path” marker
pq	Each edge with source *base* is added.

These are all appropriate for the algorithm (i.e., satisfy the conditions).

Polling the queue

When we poll the queue, we consider the target, t, of the edge removed. There are two cases:

- Its weight is already known, in which case we continue.
- Its weight is not yet known, in which case we assign it.

We need to check that both these are appropriate actions. To do this we proceed by induction on the number of keys in *weight* and add the condition that, if there are k keys in *weight* then these are the k least-weight neighbours of *base*.

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So that can't happen and it's safe to just continue.

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Huzzah!