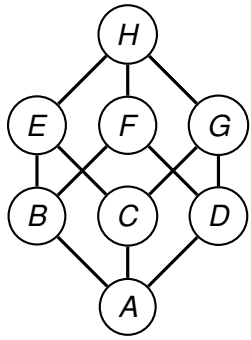
# Cosc 201
# Algorithms and Data Structures
# Lecture 11 (31/3/2025)
# Graphs: terminology and traversals

Brendan McCane
brendan.mccane@otago.ac.nz
And
Michael Albert

UNIVERSITY OF OTAGO
Te Whare Wānanga o Otāgo
NEW ZEALAND

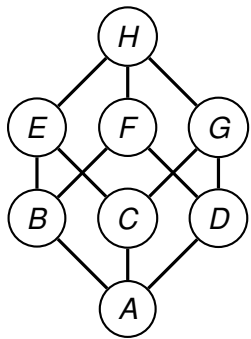
# What is a graph?

- Possibly not what you're thinking.
- A graph represents a set of things, with some relationships between pairs of them.
- It's another "data concept" because there is an enormous range of possible interpretations and conditions, and no clear interface.
- We'll be looking (mostly) at one of the simpler interpretations. It's actually called *simple graph* in mathematics.
- That's this one.



# What is a simple graph?

- A *simple graph* (just *graph* for the rest of this lecture) consists of:
  - a set of *vertices* (*A* to *H*),
  - some *edges* between them (there are 12, *AB*, *AC*, etc.)
- The picture is just a help
- And, usually, the labels are just for convenience
- Some graphs even have names like $Q_3$.



# Applications of graphs

Graphs are used to model many things, including:

- Social networks (Facebook, LinkedIn, etc.)
- Road networks (Google Maps, etc.)
- The internet (Google, etc.)
- Molecular structures (drug design, etc.)
- Scheduling (timetables, airlines, etc.)
- The power grid
- Data communications (computer networks, cell phone networks, etc.)
- Scene graphs in computer graphics
- Neural networks
- Game states (chess, etc.)
- etc

# Graph definition and terminology

- A graph $G$ is a pair $(V, E)$ where $V$ is a set of vertices and $E$ is a set of edges.
- An edge is a pair of vertices, $e = (v, w)$.
- We say that $v$ and $w$ are *adjacent* or *neighbours* if there is an edge between them.
- The *degree* of a vertex is the number of edges incident on it.
- A *simple graph* has no self-loops (edges from a vertex to itself) and no multiple edges (more than one edge between the same pair of vertices).
- A *path* is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1})$ is an edge for $1 \leq i < k$.
- A *cycle* is a path where $v_1 = v_k$.
- A *connected graph* is one where there is a path between every pair of vertices.

# Data structures to represent simple graphs

- There is no single ‘best’ way to represent graphs, even just simple graphs.
- The kinds of questions we typically want to be able to answer quickly, are things like:
  - Are vertices $v$ and $w$ *neighbours* i.e., is there an edge between them?
  - What are all the neighbours of $v$?
- The simplest way to represent a graph is to have an array of vertices, and each vertex has a list of its neighbours.
- Determining if $v$ and $w$ are neighbours is $O(\text{degree}(v))$.
- We could also have an array of edges, and each edge is a pair of vertices (no explicit representation of vertices).
- Determining if $v$ and $w$ are neighbours is $O(|E|)$.

# More complex representations

- More efficient would be an array with an element for each vertex, and each element of the array is a `HashSet` of its neighbours.
- If we wanted more structure/flexibility we could define a `Vertex` class and use a `HashMap` from `Vertex` to `HashSet<Vertex>` to represent the neighbours.
- And lots of other options too – so we'll be pretty agnostic about the details and treat the graph as an abstract data type.

# Questions we might want to answer about graphs

- Are two vertices connected?
- What is the shortest path between two vertices?
- What is the shortest path between a vertex and all other vertices?
- What groups of vertices are connected?
- What is the minimum connected subgraph that includes all vertices?
- What is the maximum flow through the graph?

We will look at some, but not all, of these questions.

# Traversals of simple graphs

- In a simple graph $G$, given vertices $v$ and $w$ we say they are *connected* if there is some sequence of edges that we could follow to get from $v$ to $w$.
- Or, in a union-find instance where we put all pairs of neighbouring vertices in the same group, $v$ and $w$ belong to the same group! That group is called the *connected component* of $v$. This is the idea used in assignment 1!
- If we think of edges as representing interaction of some kind, then vertices in different components don't influence one another, so we often implicitly assume the graph is *connected* i.e., has only one component.
- The resident of $v$ wants to visit all the vertices in their component. How might they do that?
- The *wide searcher* concentrates on visiting all the immediate neighbours, then all their (unvisited) neighbours, then all theirs, ...
- The *deep searcher* concentrates on visiting a neighbour, then one of their neighbours, then one of theirs, ...
- These strategies define the two fundamental *traversals* in graphs.

# Breadth-first traversal

We want to traverse starting from $v$. We initialise an empty queue, $q$, and add a *seen* marker to each vertex (or make a set of *seen* vertices).

```
1:  q.add(v), v.seen ← true
2:  while q is not empty do
3:      w ← q.remove()
4:      visit(w)
5:      for n in w.neighours do
6:          if n.seen = false then
7:              q.add(n)
8:              n.seen ← true
9:          end if
10:     end for
11: end while
```

- (1) Add the start vertex to a queue
- (2) While the queue is not-empty.
- (3) Remove from the front of the queue.
- (4) Do something with the item (visit)
- (5-10) Look at all its neighbours and add any unseen ones onto the queue, marking them as seen.
- (11) Repeat.

# Depth-first traversal

We want to traverse starting from $v$. We initialise an empty stack, $s$, and add a *seen* marker to each vertex (or make a set of *seen* vertices).

```
1:  s.push(v), v.seen ← true
2:  while s is not empty do
3:      w ← s.pop()
4:      visit(w)
5:      for n in w.neighours do
6:          if n.seen = false then
7:              s.push(n),
8:              n.seen ← true
9:          end if
10:     end for
11: end while
```

- (1) Add the start vertex to a stack
- (2) While the stack is not-empty.
- (3) Pop off the top element of the stack.
- (4) Do something with the item (visit)
- (5-10) Look at all its neighbours and push any unseen ones onto the stack, marking them as seen.
- (11) Repeat.

# Recursive DFT

Stack-based algorithms are often more easily expressed recursively.

```
function RECDFT(v)
    v.seen ← true, visit(v)
    for n in v.neighbours do
        if n.seen = false then
            RECDFT(n)
        end if
    end for
end function
```

# The cost of traversal

- In graph algorithms, we usually measure complexity both in terms of the number of vertices, $V$, and the number of edges $E$.
- The two traversal algorithms are so similar that one analysis will do for both.
- We assume that a single *push*, *pop*, *add* or *remove* is $O(1)$.
- What happens to each vertex?
  - It occurs in some neighbour list for the first time at which point it is added to the queue or stack, and marked as seen (6-7).
  - Then at some future point it is removed from the stack or queue (3).
  - That's $O(1)$ work per vertex, so $O(V)$.
- Where do the edges come in?
  - The edges arise implicitly in the neighbour lists (4).
  - Each one appears twice, once per endpoint.
  - When one appears, it causes a constant amount of work to happen (5).
  - That's $O(1)$ work per edge, so $O(E)$.
- The total complexity is $O(V + E)$.